Introduction to the Stochastic Growth Model

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Structure of the Seminar

⋆ Day 1

1. Introduction to Discrete Stochastic Processes
2. Deterministic Dynamics
3. Stochastic Dynamics via Markov Operators

⋆ Day 2

4. Stability of Markov Processes
5. Applications
6. Empirics of Stochastic Growth

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Part 1: Discrete Stochastic Processes

- ⋆ Choice under uncertainty
- ⋆ Finite horizon control
- ⋆ Extending to the infinite horizon
- ⋆ Introduction to stochastic growth

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Choice under uncertainty

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According to this we can rank actions.

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This induces an order on space of actions G.

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Total reward is

$$\mathbb{E}\left[\sum_{t=0}^{T-1} \beta^t r(x_t, u_t) + w(x_T)\right]. \tag{3}$$

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Since h and the distribution of ε_0 is known, for each g we can calculate the conditional distribution of x_1 given x_0. Call it $f_g(x_1|x_0)$.

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Then $F_g \in \mathscr{P}_S$ for all g.

Also, let v_g be the real function on $S = X \times X$ defined by

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Problem is then

$$\max_g \mathbb{E}(v_g | F_g) = \max_g \int_S v_g(s) F_g(ds). \tag{10}$$

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$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t r(x_t, u_t)\right] \tag{13}$$

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The problem is stationary, so choose just one g.

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Making suitable analogous definitions, they show that all of the results of the deterministic case carry over.

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Each g selects a joint distribution F_g over $\mathbb{R}^{\mathbb{N}}$. Let v_g be the function $\sum_t \beta^t u(g(x_t))$ from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$. Solve

$$\max_g \int_S v_g(s) F_g(ds). \tag{19}$$

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Theorem (Mirman–Zilcha). An optimal consumption policy g exists and is unique. It satisfies

$$u'(g(x)) = \beta \int u'[g(f(x - g(x))z)] f'(x - g(x)) z \psi(dz) \tag{20}$$

for all $x \geq 0$.

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Taking logs gives the linear-Gaussian AR(1) system

$$\tilde{x}_{t+1} = a\tilde{x}_t + b + \tilde{\varepsilon}_t, \quad a < 1. \tag{22}$$

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Part 2: Deterministic Dynamics

- ⋆ Semidynamical systems
- ⋆ Stability and equilibrium
- ⋆ Brouwer-Schauder fixed point theorem
- ⋆ Banach contraction theorem
- ⋆ Contractions and compactness
- ⋆ Lagrange stability and contractions

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Thus $x_1 = g(x_0)$, $x_2 = g(g(x_0)) \equiv g^2(x_0)$,..., $x_t = g^t(x_0)$.

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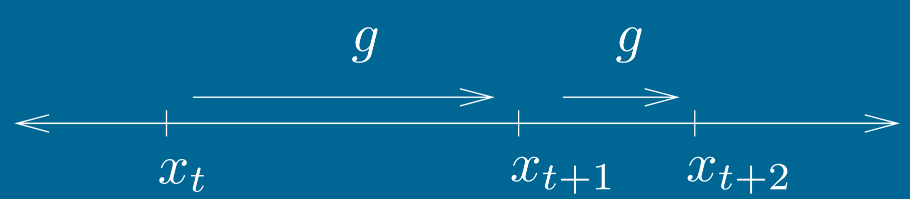

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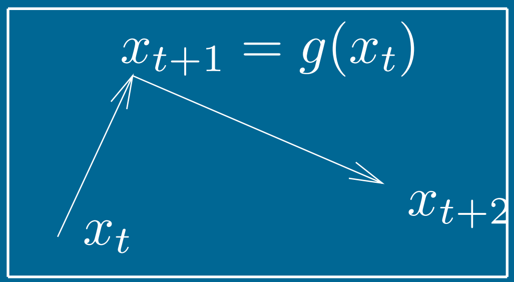

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Stability and equilibrium

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Definition. Equilibrium x^* is called *stable* if $S_g(x^*)$ contains an open neighborhood of x^*.

Definition. It is *globally stable* if $S_g(x^*) = X$.

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Brouwer-Schauder fixed point theorem

Theorem. Let X be a subset of a normed linear space. If X is compact and convex, and, in addition, g is continuous on X, then (X, g) has at least one equilibrium.

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Banach contraction theorem

Definition. Let (X, g) be a semidynamical system, with X a subset of a normed linear space. Map g is called a *Banach contraction* if there exists a $\lambda < 1$ such that

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Theorem. If X is complete and g a Banach contraction, then (X, g) has a unique and globally stable equilibrium.

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The fixed point is unique.

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Contractions and compactness

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Note that strongly contracting does not imply existence of f.p. (e.g. speed is $1 + 1/n$).

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Convergence: Not proved, but true.

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Note that Lagrange stability substitutes for compactness.

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Note that $\gamma(x)$ is compact.

Also, g maps $\gamma(x)$ into $\gamma(x)$.

Note that g is contracting on $\gamma(x)$.

Hence a fixed point exists in $\gamma(x)$.

Part 3: Stochastic Dynamics via Markov Operators

- ⋆ The general Markov operator
- ⋆ Construction from perturbed systems
- ⋆ Equilibrium and stability
- ⋆ Example: the AR(1) model

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The general Markov operator

We want to study something like this:

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How can we apply techniques for deterministic systems?

The method: transform this into deterministic system on an infinite dimensional space called $\mathscr{L}_1$.

As before, X is some space. Transform X into a measure space $(X, \mathscr{B}_X, \mu)$.

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The deterministic model is a semidynamical system in X, but the stochastic version is a semidynamical system in $\mathscr{L}_1(\mu)$!

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A *Markov operator* is an operator $P \colon \mathscr{L}_1(\mu) \to \mathscr{L}_1(\mu)$ such that

$$f \in \mathscr{D}(\mu) \implies Pf \in \mathscr{D}(\mu). \tag{28}$$

Construction from perturbed systems

Consider the (very common macroeconomic) model

$$x_{t+1} = g(x_t, \varepsilon_t). \tag{29}$$

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We can construct a *conditional distribution* $\Gamma(x_{t+1}, x_t)$ for x_{t+1} given x_t and knowledge of g, ψ.

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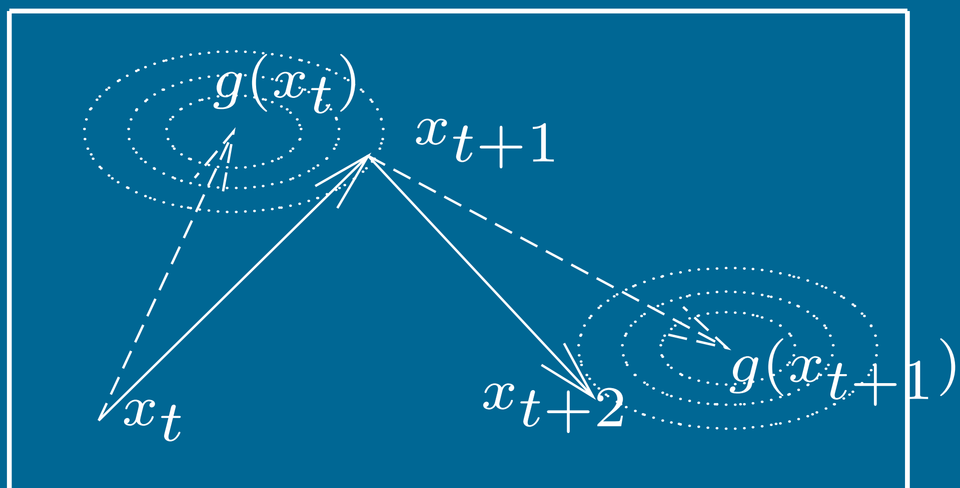

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And similarly,

$$\varphi_2(x_2) = \int \Gamma(x_1, x_2)\varphi_1(x_1)dx_1. \tag{31}$$

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So now define an operator P mapping $\mathscr{L}_1(\mu)$ into itself, where if f is in $\mathscr{L}_1(\mu)$ then $Pf \in \mathscr{L}_1(\mu)$ is defined by

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Therefore $(\mathscr{D}(\mu), P)$ is a semidynamical system!!

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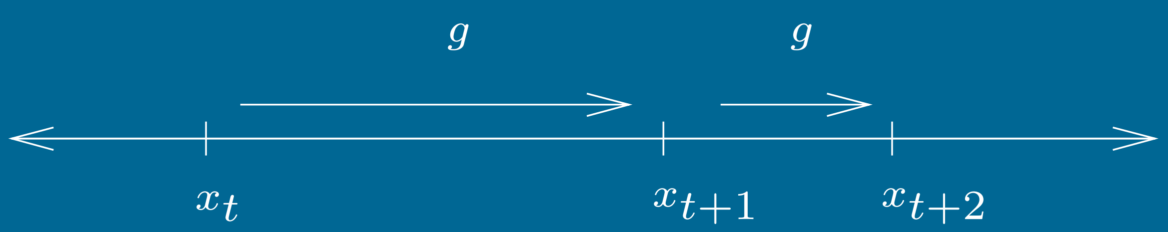

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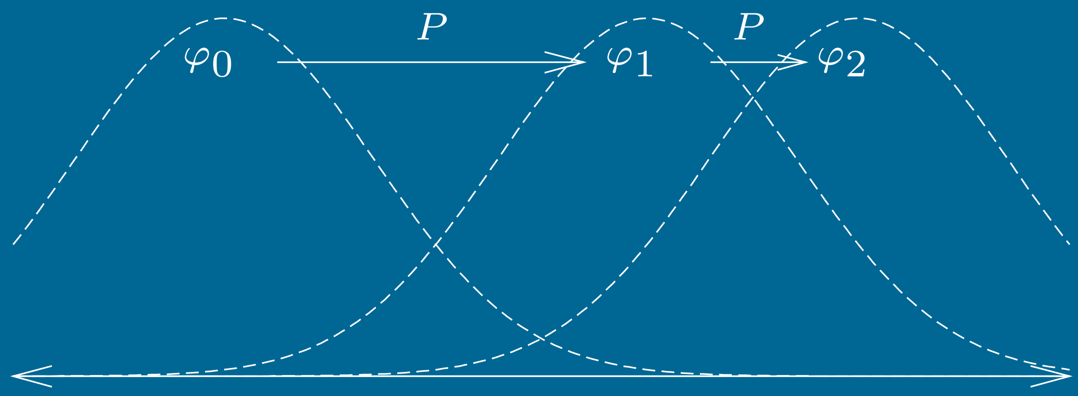

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Equilibrium and stability

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But this just means a $\varphi^* \in \mathscr{D}(\mu)$ s.t. $P\varphi^* = \varphi^*$.

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Part 4: Stability of Markov Processes

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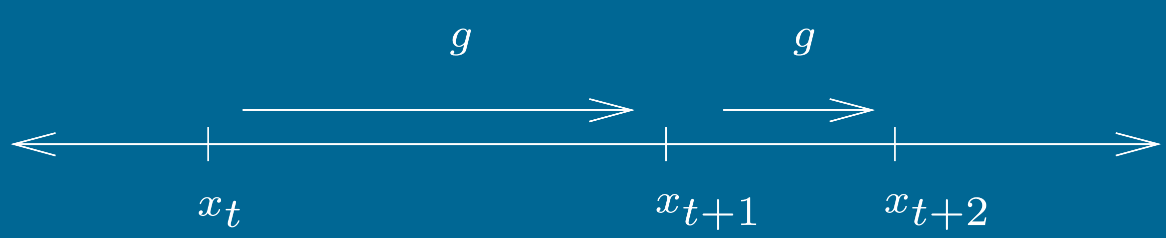

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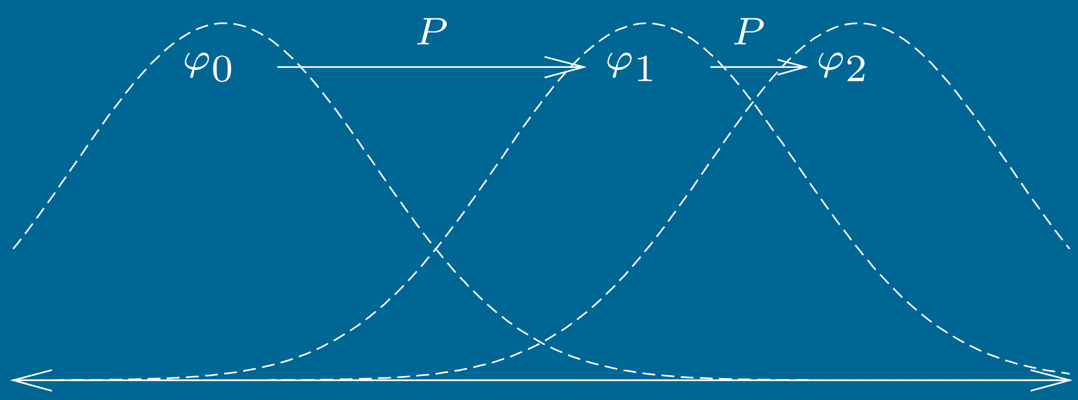

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and $\{P^t \varphi\}$ is precompact, $\forall \varphi \in \mathscr{D}(\mu)$, then exists unique, globally stable equilibrium

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Strongly contracting Markov operators

Proposition. If $\Gamma(x, y) > 0$ for all x, y, then P is strongly contracting.

Intuition...

Strongly contracting Markov operators

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Intuition... I don't know

Proof...

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$$\|P\varphi' - P\varphi''\| = \|P(\varphi' - \varphi'')\|$$

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$$= \int |\int \Gamma(x, y)[\varphi'(x) - \varphi''(x)]dx|dy$$

$$
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Corollary. At most one equilibrium

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$\mathscr{D}_0$ is *tight* if $\forall \varepsilon > 0$, $\exists K$ compact such that $\int_{K^c} \varphi < \varepsilon$ whenever $\varphi \in \mathscr{D}_0$.

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Proposition. Let $X = \mathbb{R}$, and let

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Let P be the associated Markov operator. If $|a| < 1$, then $P^t \varphi$ is tight for every Gaussian $\varphi \in \mathscr{D}(\mu)$.

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Chebychev inequality: If ξ is an r.v. on $\mathbb{R}$, then

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Therefore, $a < 1$ implies tightness

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Part 5: Applications

- ⋆ The stochastic growth model
- ⋆ Stability in the increasing returns model

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The stochastic growth model

Recall that problem is to choose a consumption policy $g\colon x \longrightarrow c$ to maximize

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Recall that if optimal g exists then

$$u'(g(x)) = \beta \int u'[g(f(x - g(x))z)] f'(x - g(x)) z \psi(dz). \tag{46}$$

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Under these assumptions g exists and is unique.

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Unbounded shock: Stachurski (JET, 2002)

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And hence P as per usual

$$Pf(y) = \int \Gamma(x, y) f(x) dx.$$

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Note: This is all we need for uniqueness.

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$$\psi(z) = (2\pi\sigma^2)^{-1/2}\frac{1}{z}\exp\left[-\frac{(\ln z - \mu)^2}{2\sigma^2}\right]. \tag{52}$$

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Same for many common shocks.

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Evidently following two conditions are sufficient

$$\forall \varepsilon > 0,\ \exists r > 0 \ \text{ s.t. } \left\{ \int_r^\infty P^t \varphi(x) dx < \varepsilon, \quad \forall t \in \mathbb{N}_0 \right\} \tag{53}$$

and

$$\forall \varepsilon > 0,\ \exists r > 0 \ \text{ s.t. } \left\{ \int_0^{1/r} P^t \varphi(x) dx < \varepsilon \quad \forall t \in \mathbb{N}_0 \right\} \tag{54}$$

if we set $K = [1/r, r]$, r sufficiently large.

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We prove only the first condition:

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need show only that the sequence of real numbers $\mathbb{E}_\varphi x_t$ is bounded.

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$$\begin{aligned}
\mathbb{E}_\varphi x_t &= \int_0^\infty \mathbb{E}(x_t | x_{t-1} = x) \, \mathrm{Prob}(x_{t-1} = x) dx \\
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This completes proof of global stability!

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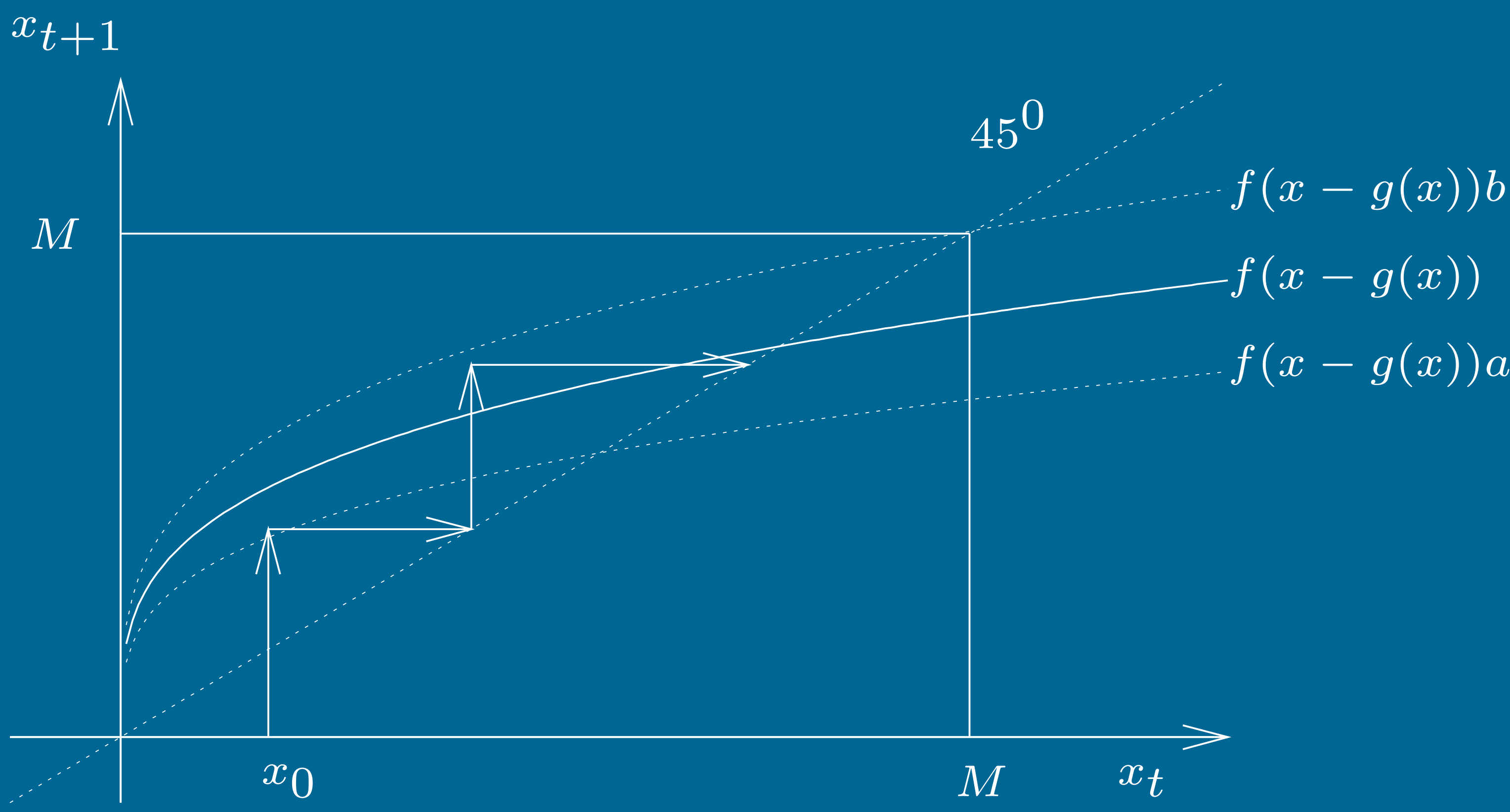
By working a bit harder for tightness we can incorporate models with unbounded state.



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Stability in the increasing returns model

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Depreciation is total.

The production function is

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Note nonstationarity of technology as a result of A_t.

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Let $f(k) = F(k, 1)$.

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Assumption. The function $f\colon [0, \infty) \to [0, \infty)$ satisfies $f(0) = 0$, $f' > 0$, $f'' < 0$, $f'(0) = \infty$, $f'(\infty) = 0$.

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In the overlapping generations model, additional assumptions are required to prevent the economy collapsing to zero output.

Assumption. The inequality $kf'(k) \leq \lambda f(k)$ holds everywhere for some $\lambda < 1$.

This says that the capital share of income cannot become arbitrarily close to total income.

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Optimization:

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Assumption. consumer maximize

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Competitive factor markets imply that $w_t = A_t[f(k_t) - k_t f'(k_t)]\varepsilon_t^\sigma$.

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Assumption. $\operatorname{rng} A(k)$ compact.

Law of motion is

$$k_{t+1} = S(k_t)\varepsilon_t^{\sigma}, \tag{56}$$

where $S(k) = DA(k)[f(k) - kf'(k)]$.

For example, the “critical mass” form

$$A(k) = A_1 \cdot \mathbf{1}_{[0,k_b)}(k) + A_2 \cdot \mathbf{1}_{[k_b,\infty)}(k),$$

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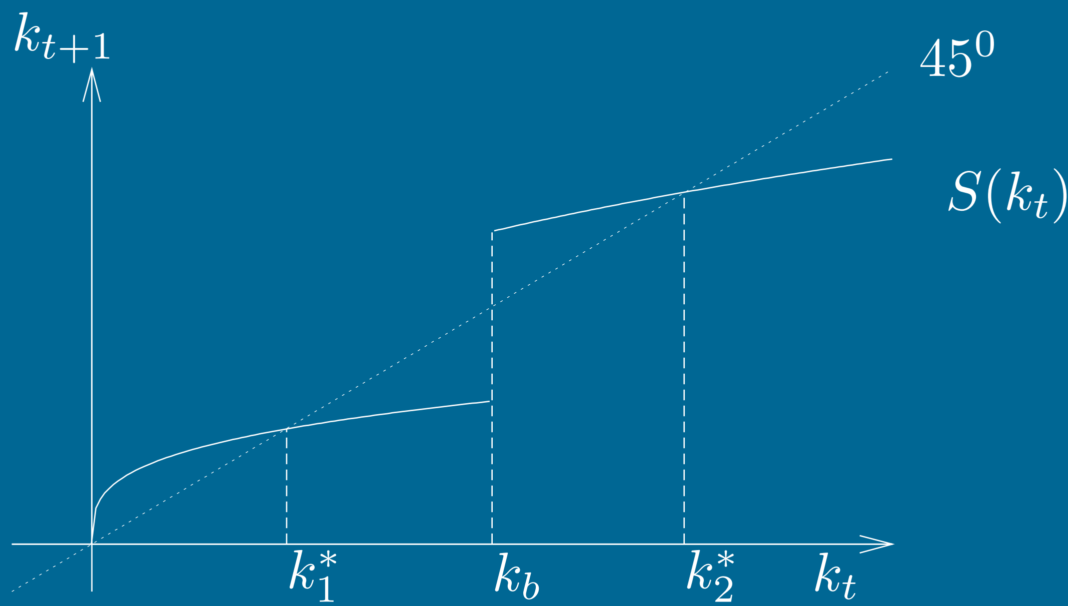

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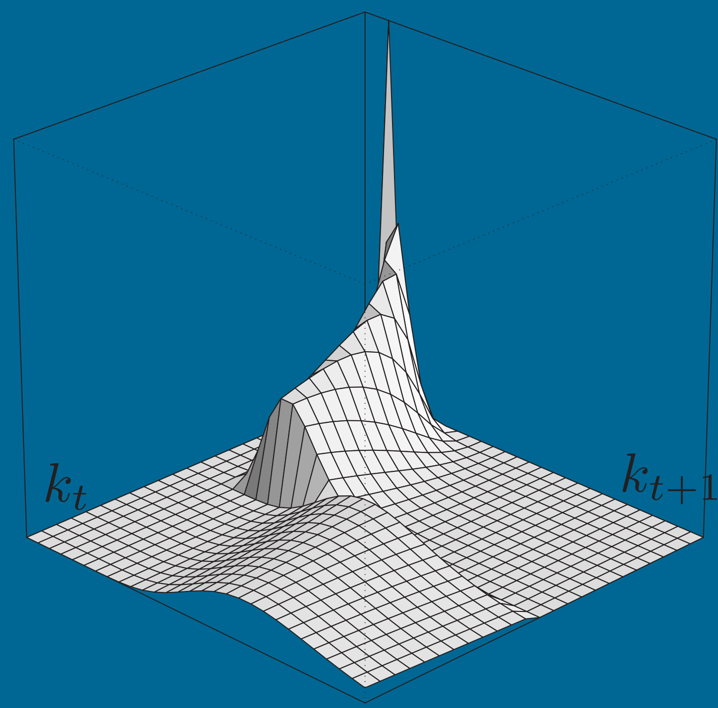

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Proposition. If $\sigma = 0$ then there may be multiple, locally stable equilibria. However, for every $\sigma > 0$, there is a single, unique (stochastic) equilibrium.

Proof: we will show that for any $\sigma > 0$, $(\mathscr{D}(\mu), P_\sigma)$ is strongly contracting and Lagrange stable.

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But if $\varepsilon \sim \psi$, then

$$\Gamma_\sigma(k, k') = \psi \left[\left(\frac{k'}{S(k)} \right)^{\frac{1}{\sigma}} \right] \left(\frac{k'}{S(k)} \right)^{\frac{1}{\sigma}} \frac{1}{\sigma k'}, \tag{58}$$

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This is all we need for uniqueness!

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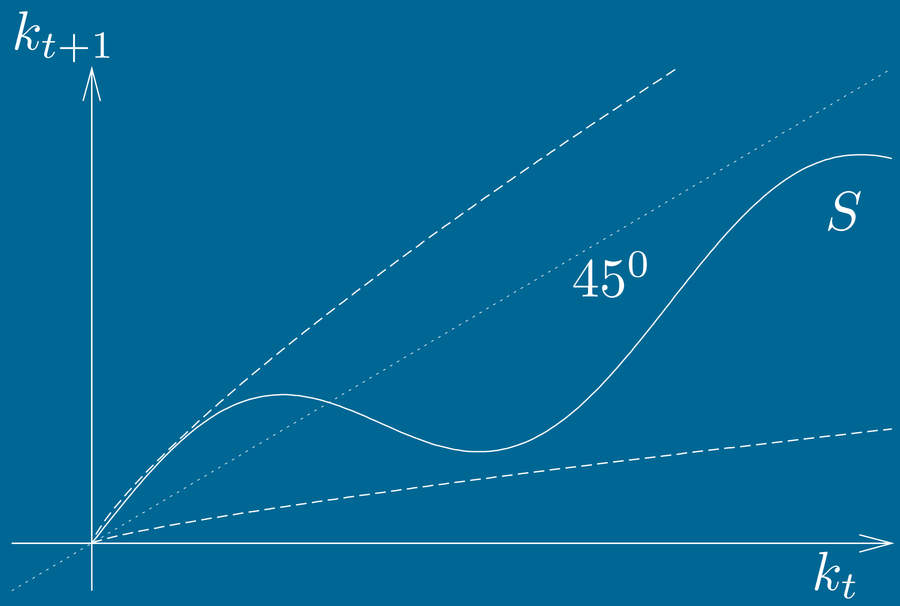
Let us assume that



More formally, given

$$k_{t+1} = S(k_t)\varepsilon_t^{\sigma}, \tag{59}$$

assume that exists α, β_i positive with $\alpha < 1$,

$$\beta_1 k^{\alpha} \leq S(k) \leq \beta_2 k^{\alpha}.$$

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Then Lagrange stable.

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Recall the

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1. exists continuous h such that $\Gamma(x, y) \leq h(y)$, and

2. $\{P^t \varphi\}$ is tight,

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Condition 1 holds again for the lognormal shock (we will not check it).

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Chebychev inequality:

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Therefore, for each t,

$$\int_{[\exp(-r),\exp(r)]^c} P^t \varphi(k) dk \leq \frac{\mathbb{E}_\varphi |\ln k_t|}{r}, \quad \forall r > 0. \tag{61}$$

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If $\exists M < \infty$ s.t. $\mathbb{E}_\varphi |\ln k_t| \leq M$ for all t then we are done.

We have

$$\mathbb{E}_{\varphi}|\ln k_t| = \int_0^{\infty} \mathbb{E}(|\ln k_t| \text{ given } \; k_{t-1} = k)\,\mathrm{Prob}(k_{t-1} = k)dk$$

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$$\leq \int_0^\infty \{|\ln S(k)| + \sigma \mathbb{E}_\varepsilon |\ln \varepsilon|\} P^{t-1}\varphi(k) dk.$$
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Note also that

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$$\beta_1 k^\alpha \leq S(k) \leq \beta_2 k^\alpha \implies |\ln S(k)| \leq \alpha |\ln k| + M.$$

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To repeat,

$$\mathbb{E}_{\varphi}|\ln k_t| \leq \int |\ln S(k)|P^{t-1}\varphi(k)dk + \sigma \mathbb{E}_{\varepsilon}|\ln \varepsilon|.$$

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Which is sufficient for the proof.

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Notice our proof works for any noise level σ.

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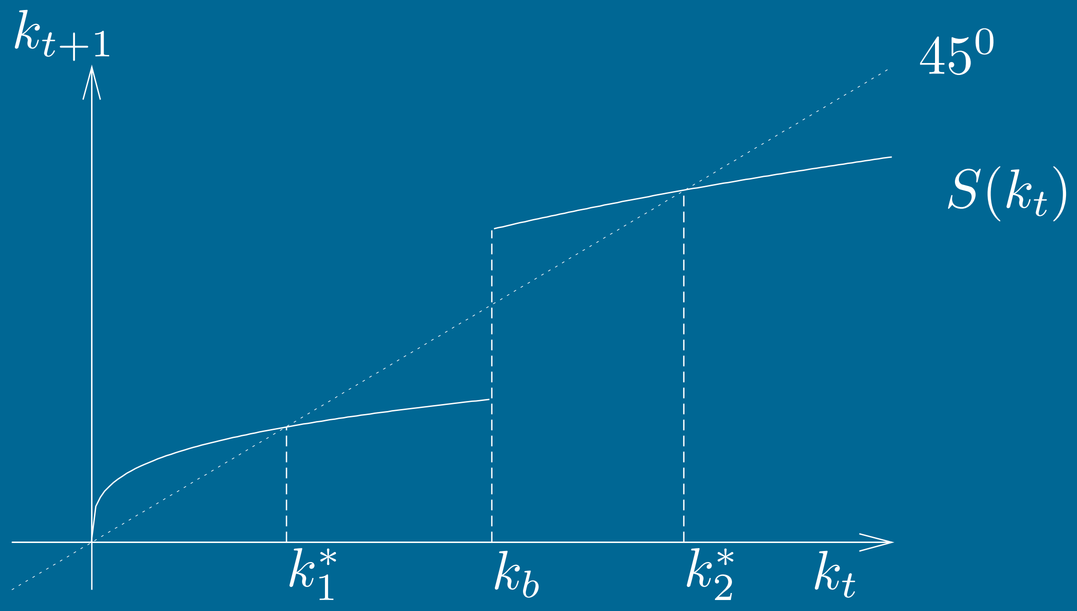
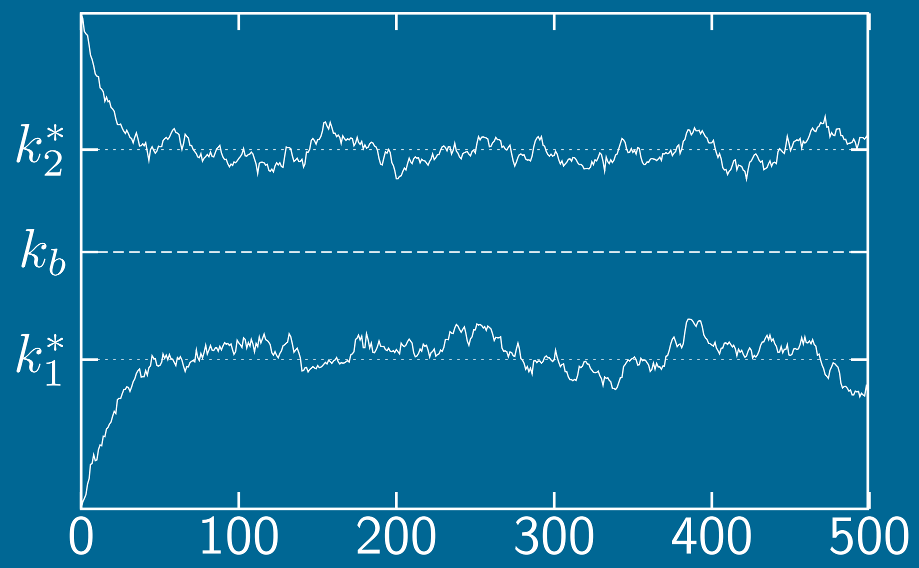
This is suprising, because when noise level is nearly zero, behavior must be similar to



Indeed this is the case: History does not matter, but



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Part 6: Empirics of Stochastic Growth

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Some stylized facts

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To do so we must choose an appropriate model.

The convex neoclassical model is rejected.

Instead, we fit and predict with the increasing returns Azariadis-Drazen model.

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Begin with stylized facts: (Lant Pritchett JEP, 11(3), 1997)

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2. What accounts for the few countries that are able to initiate and sustain periods of rapid growth in which they gain significantly on the leaders?

3. What accounts for why some countries fade and lose the momentum of rapid growth?

Begin with stylized facts: (Lant Pritchett JEP, 11(3), 1997)

1. What accounts for continued per capita growth and technological progress of those leading countries at the frontier?

2. What accounts for the few countries that are able to initiate and sustain periods of rapid growth in which they gain significantly on the leaders?

3. What accounts for why some countries fade and lose the momentum of rapid growth?

4. What accounts for why some countries remain in low growth for very long periods?

Also, we can look at the data: (countries with per capita GDP data for both 1960 and 1990 in the Summers and Heston (1991) PWT5.6)

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3. Regarding rapid take-off, some poor–middle income countries grew *much* faster than rich countries (S. Korea $\uparrow$ by factor of 7.4; Singapore, 7.1; Hong Kong, 6.6; Taiwan, 6.4; Japan, 4.9; Portugal, 4.2).

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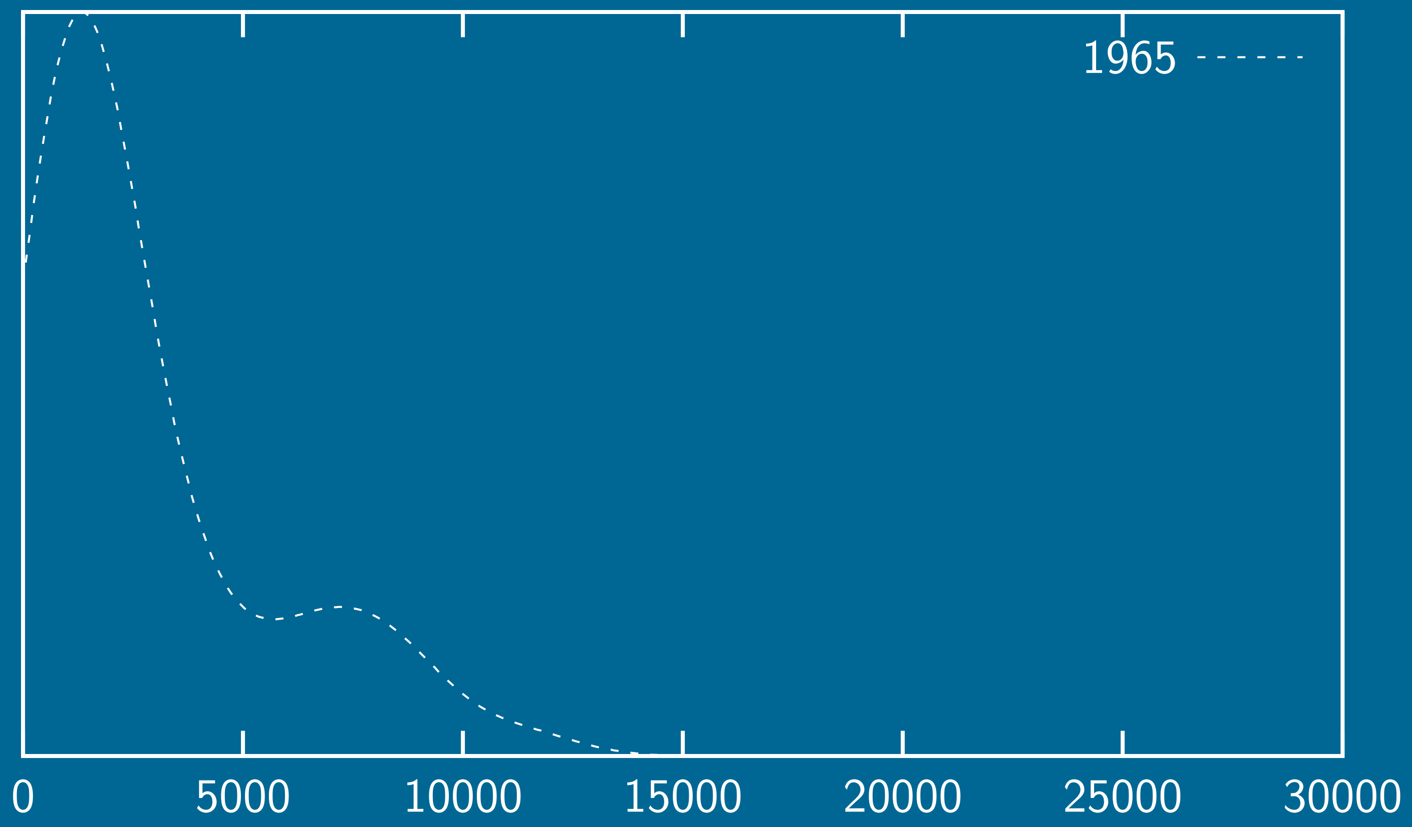

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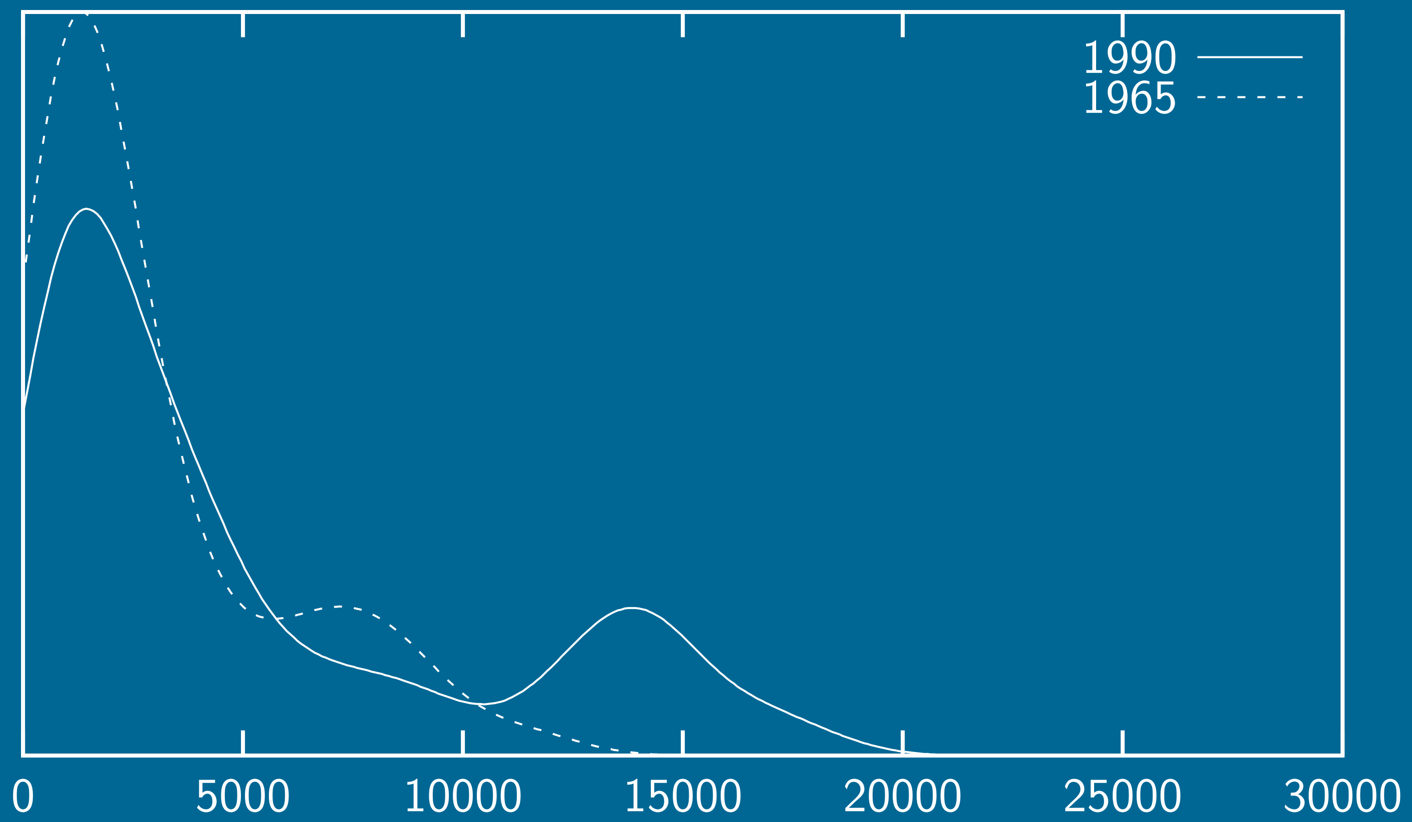

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Fitting the convex model

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Then *all of trajectory is lognormal* with parameters

$$\mu_{t+1} = \ln s + \alpha \mu_t, \quad \mu_0 = \mathbb{E} k_0. \tag{63}$$

$$\sigma_{t+1}^2 = \sigma_\varepsilon^2 + \alpha^2 \sigma_t^2, \quad \sigma_0^2 = \mathbb{V} k_0. \tag{64}$$

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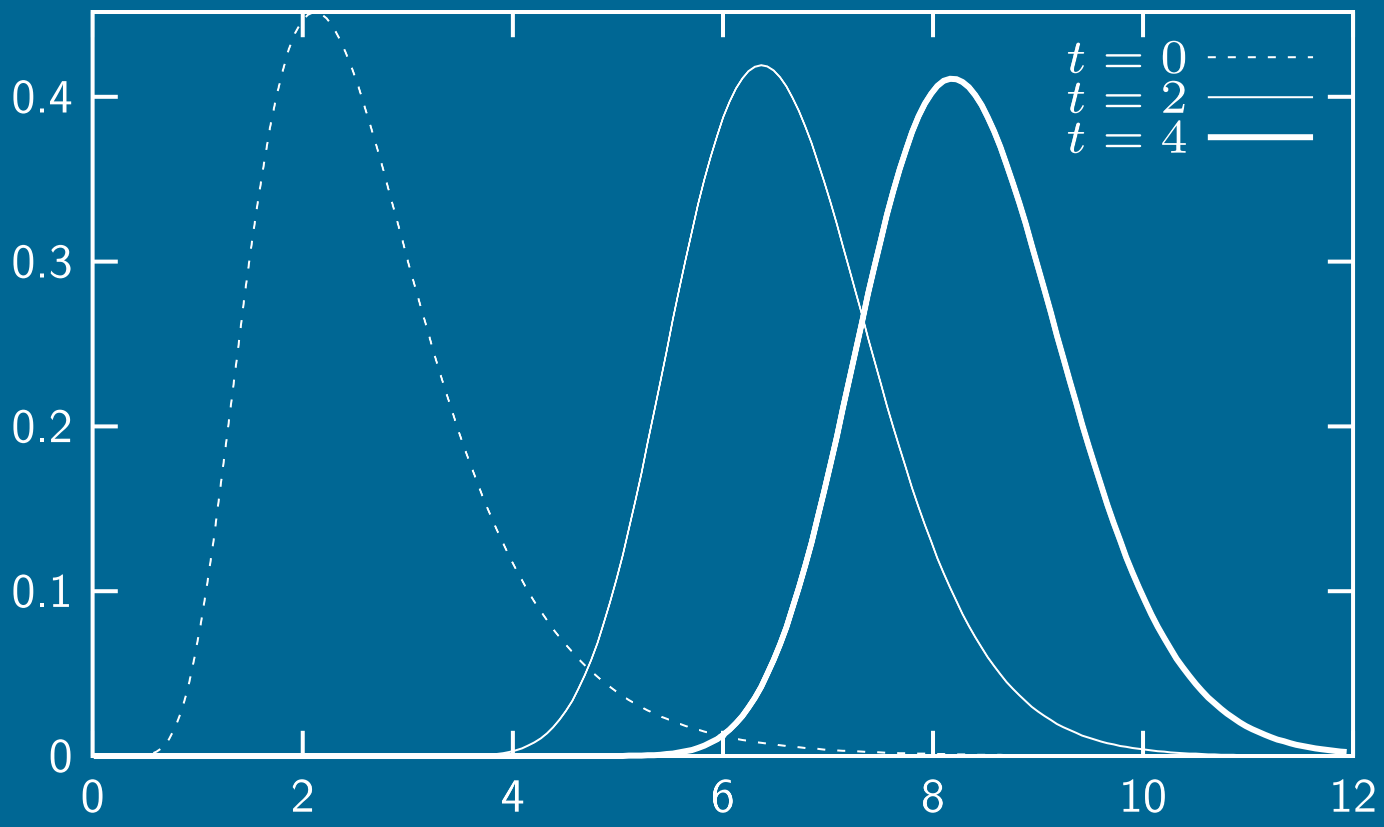

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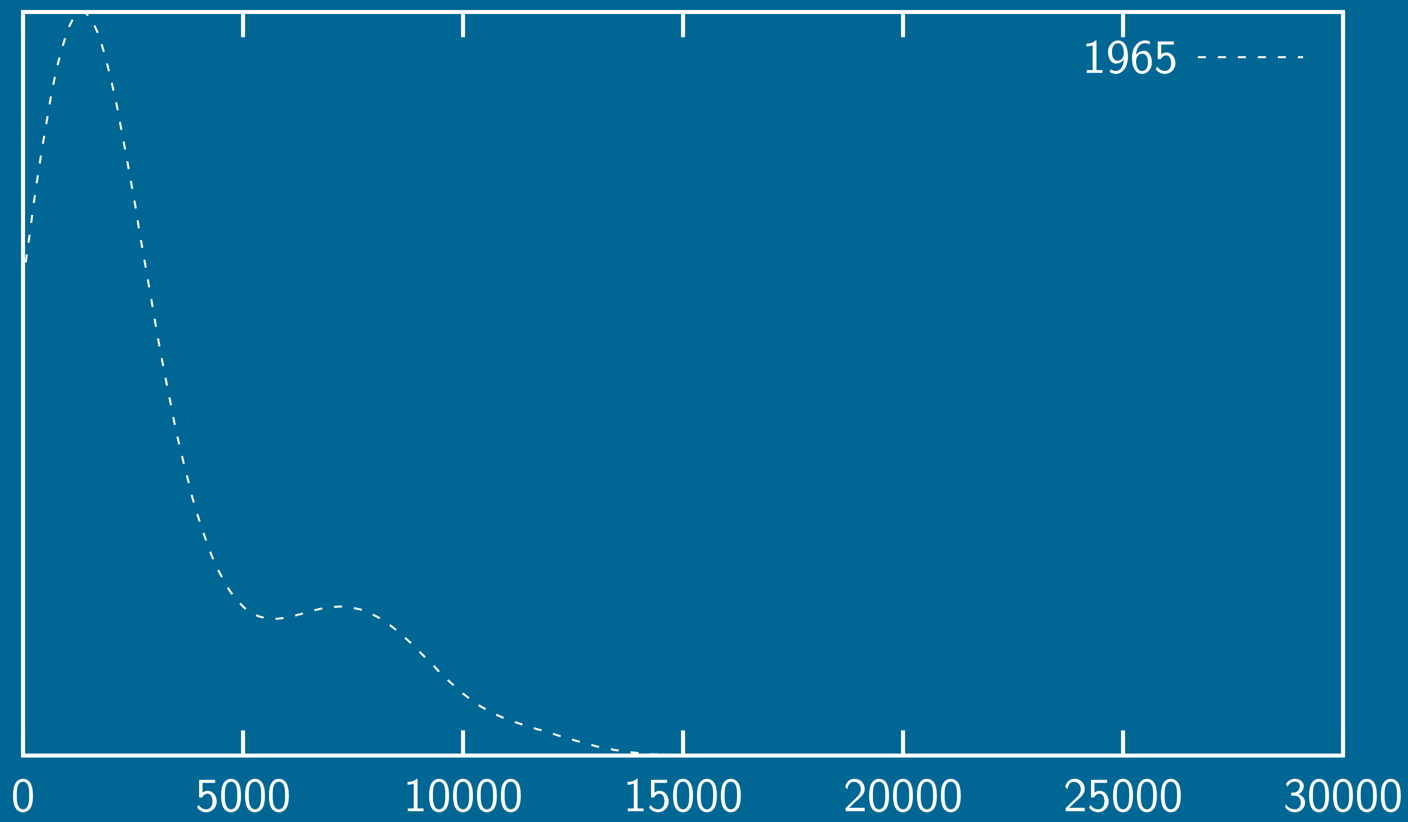

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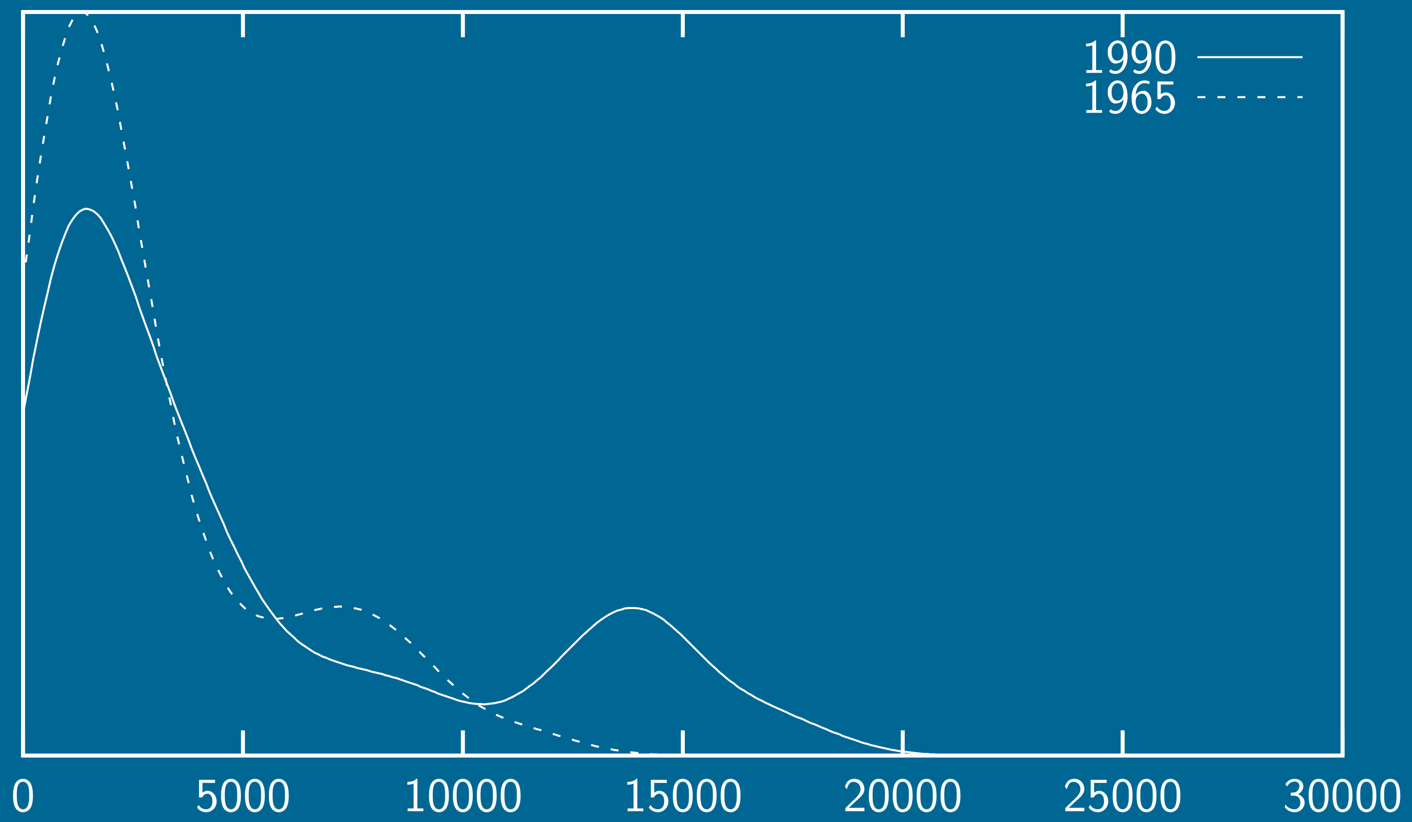

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Fitting the Azariadis-Drazen model

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Here we fit, and then use the operator P to project densities into the future.

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Recall the Azariadis-Drazen model.

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In the Cobb-Douglas case,

$$k_{t+1} = S(k_t)\varepsilon_t^{\sigma}, \tag{65}$$

where, $S(k) = D(1-\alpha)A(k)k^{\alpha}$.

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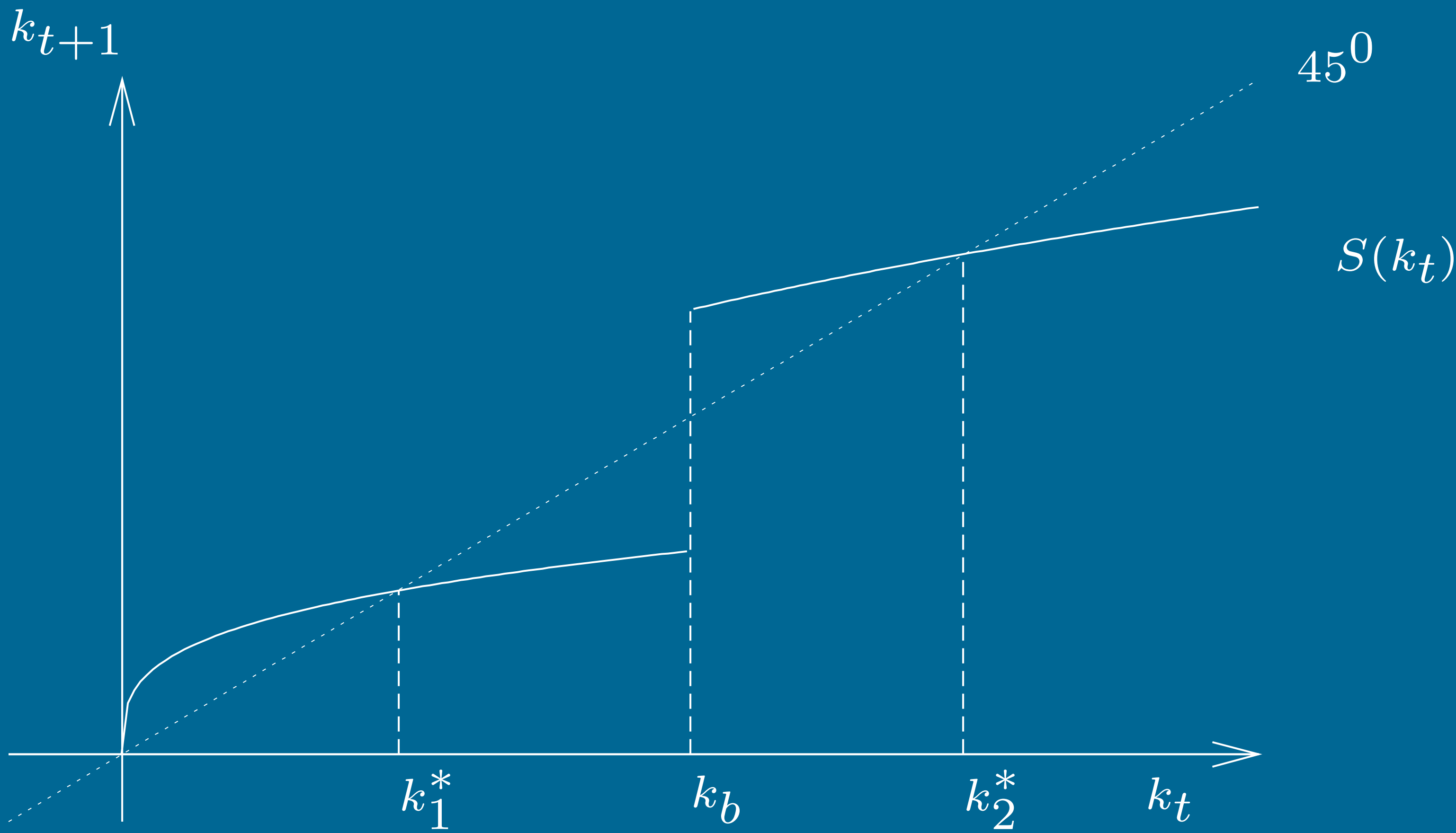

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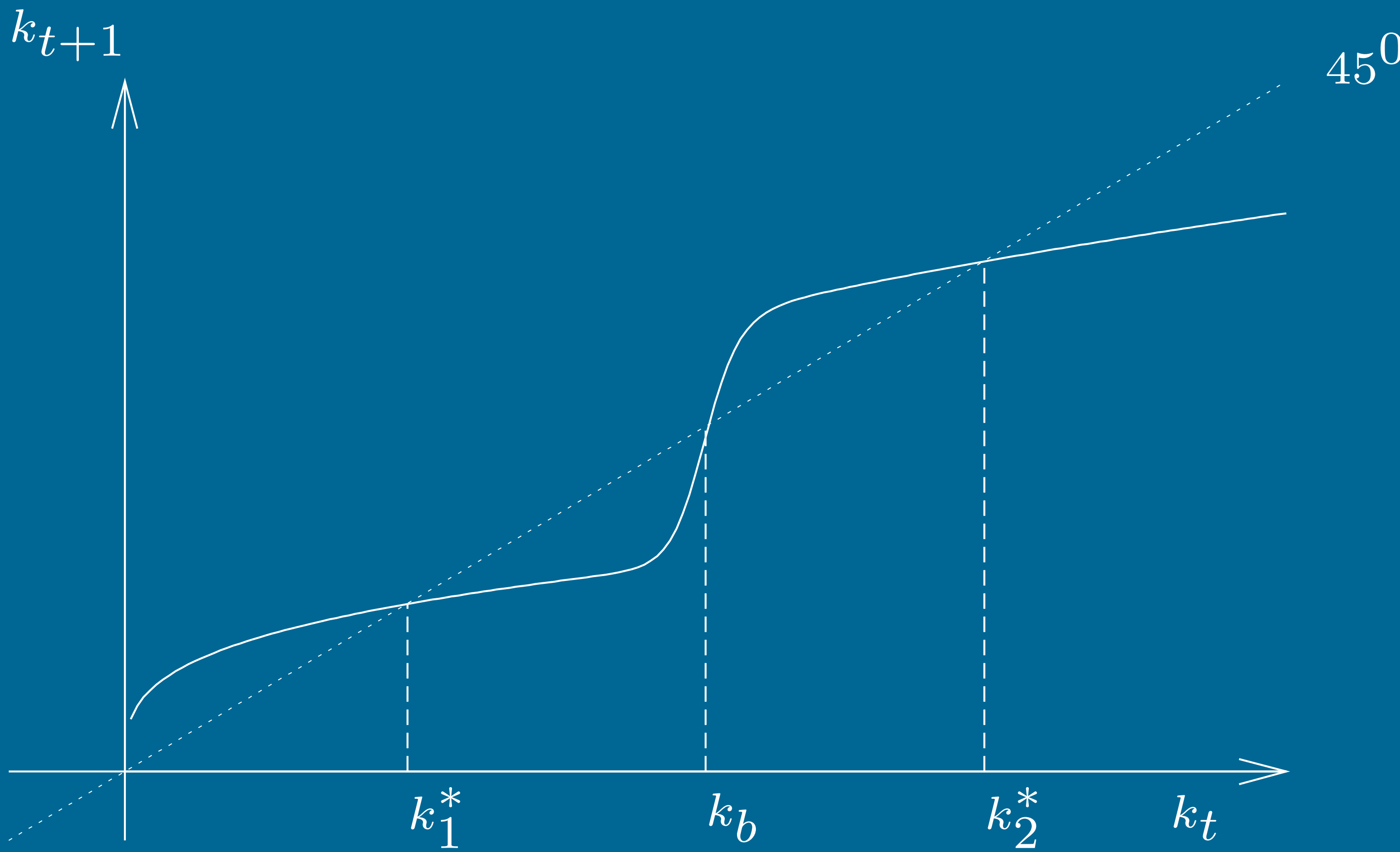

k_{t+1}
45^0
k_1^*
k_b
k_2^*
k_t

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We let $t = 1965$ and $t + 1 = 1990$ (25 years for OLG).

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Data is pooled and we regress across all countries i:

$$k^i_{1990} = S(k^i_{1965})\varepsilon^\sigma_{1965}, \quad i = 1, \ldots 105. \tag{67}$$

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The smooth transition version uses STAR.

Implementation of the procedure is in Java.

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Once the model

$$k_{t+1} = S(k_t)\varepsilon_t^{\sigma}, \tag{68}$$

is known, we also know the conditional density:

$$\Gamma_{\sigma}(k, k') = \psi\left[\left(\frac{k'}{S(k)}\right)^{\frac{1}{\sigma}}\right]\left(\frac{k'}{S(k)}\right)^{\frac{1}{\sigma}}\frac{1}{\sigma k'}, \tag{69}$$

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and hence the Markov operator

$$Pf(k') = \int \Gamma(k, k')f(k)dk.$$

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There are M countries. All of them have the same dynamics, which are known to us.

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How should we guess the distribution of y_m^1?

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$$P\hat{\varphi}_0 \to P\varphi_0 = \varphi_1.$$

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This gives 75 year and 125 year projections for the cross-country income distribution.

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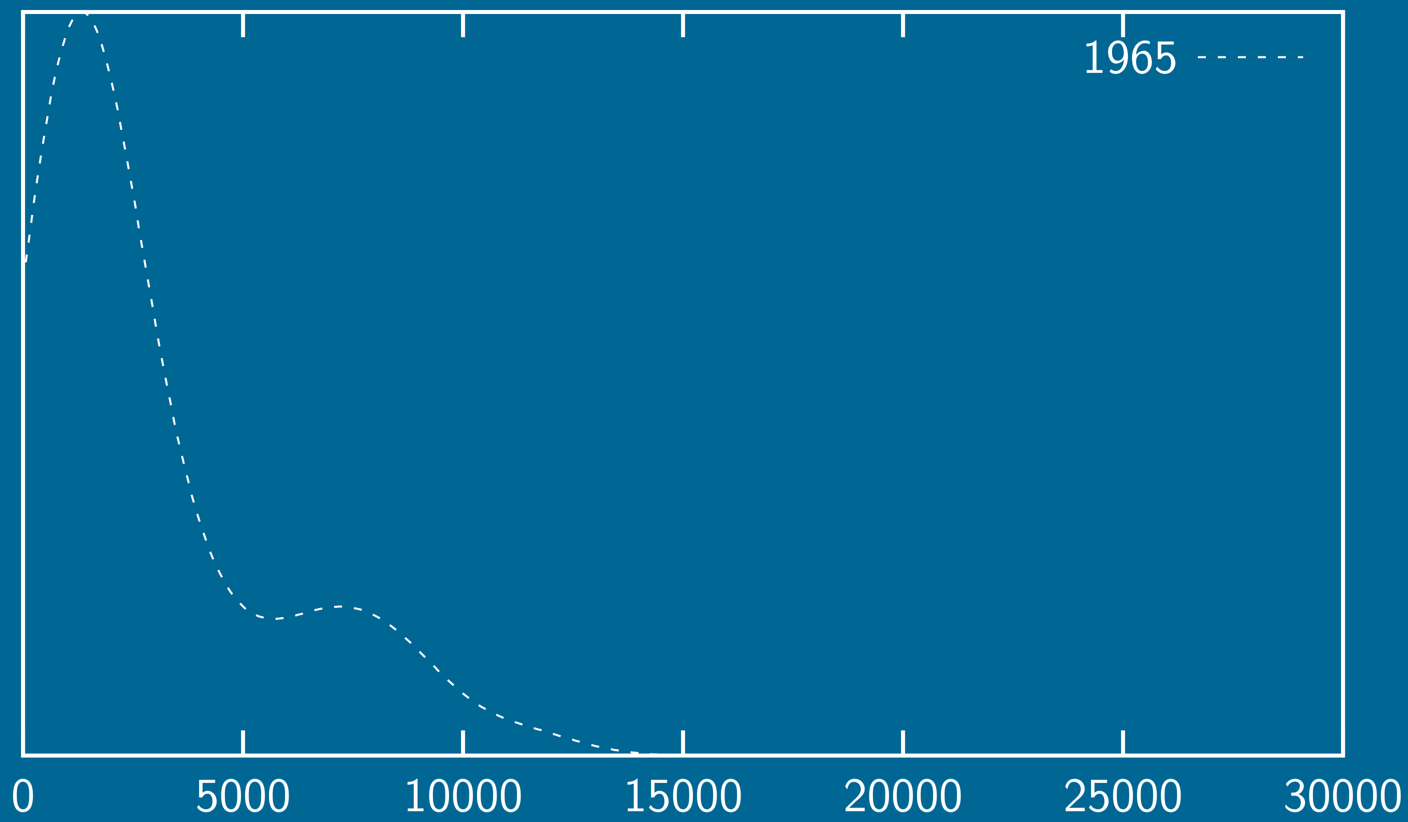

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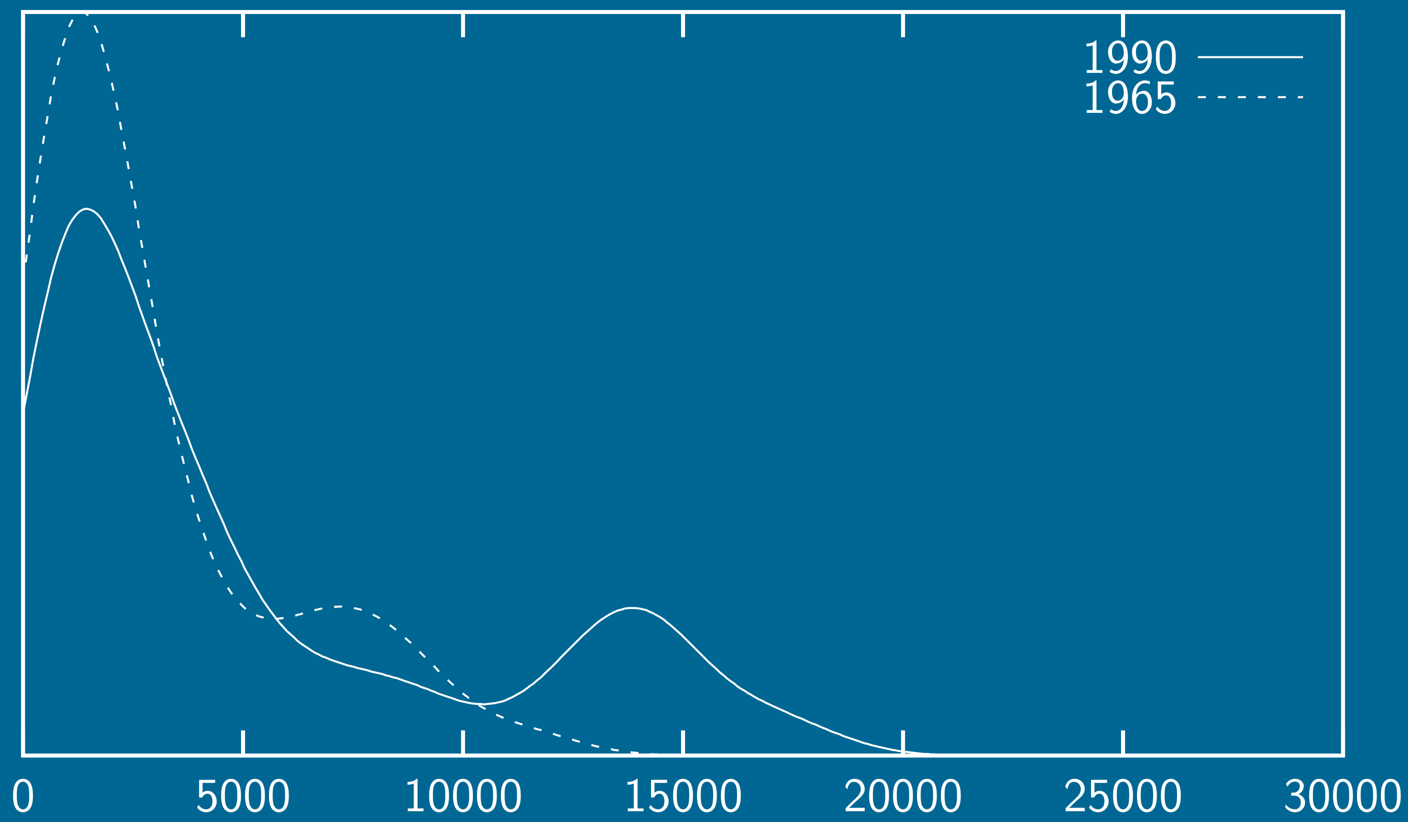

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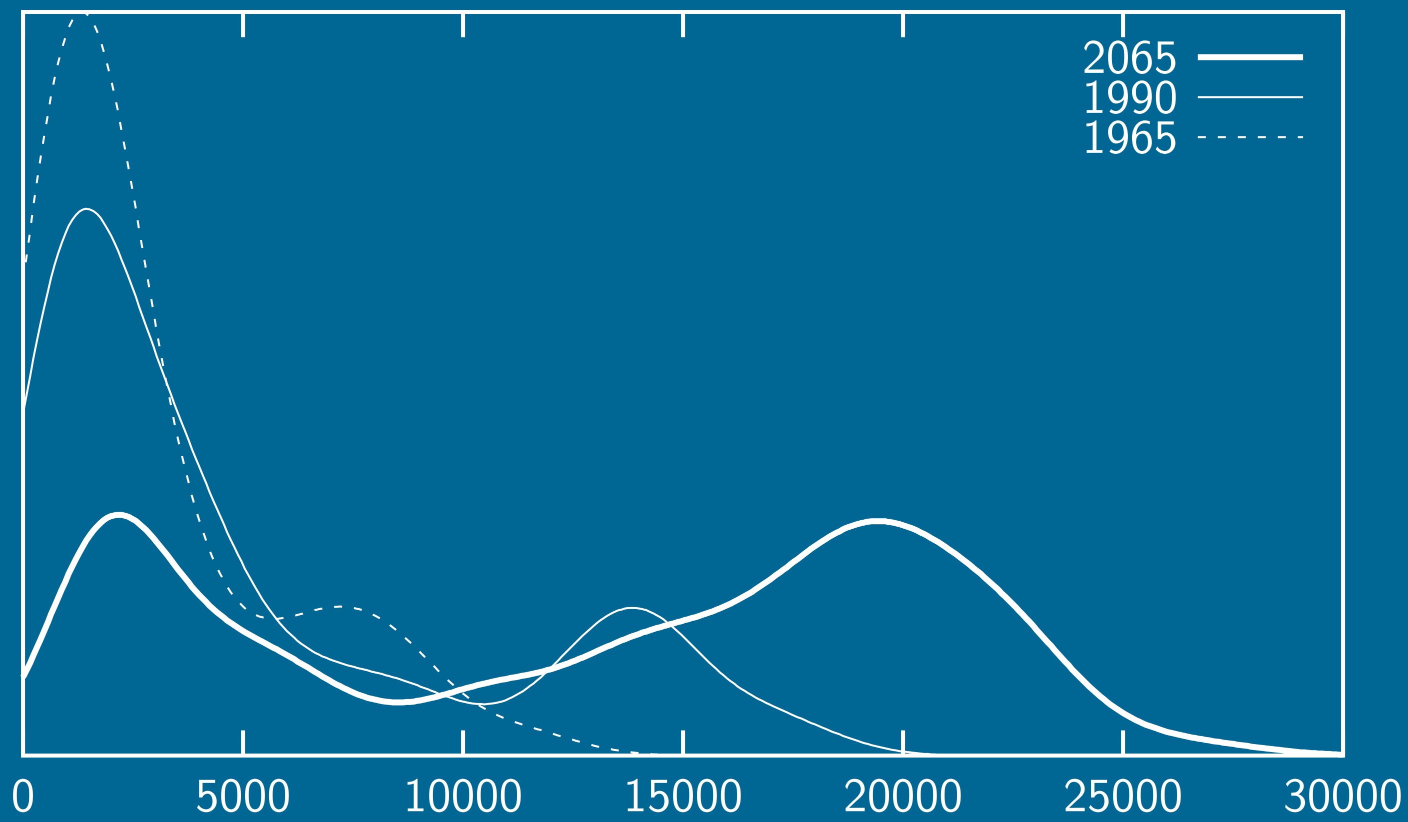

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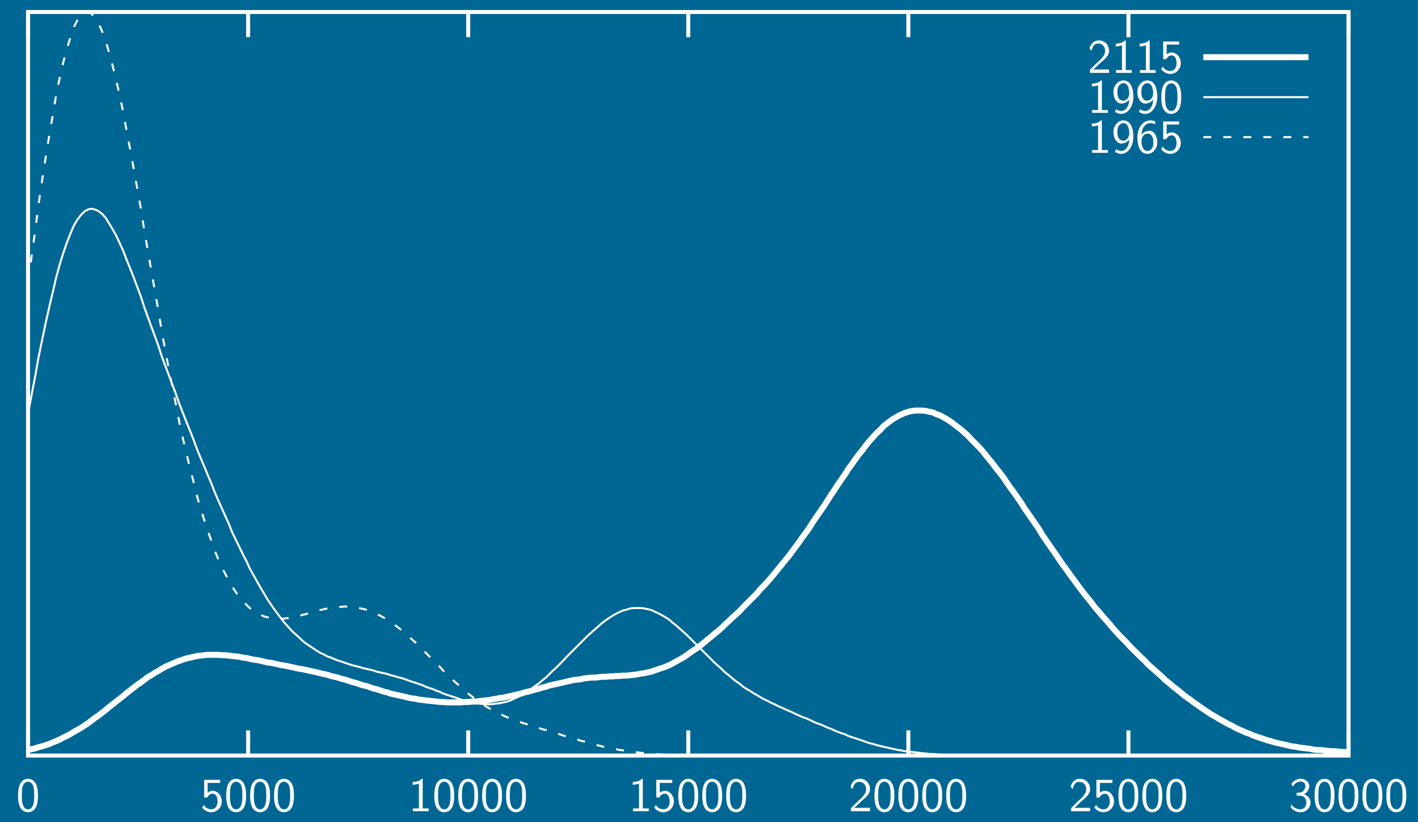

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Kuznets hypothesis: inequality $\uparrow$ and then $\downarrow$ over time as industrialization occurs.

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Gini coefficient:

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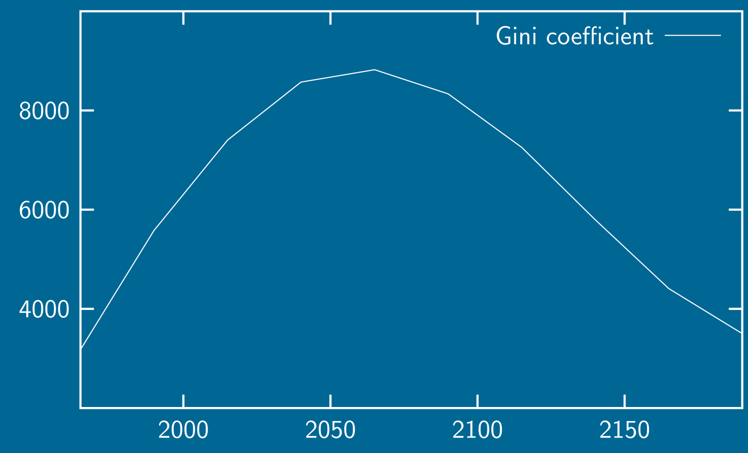

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